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The normalizer property for integral group rings of Frobenius groups

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Abstract

We prove that the normalizer property holds for the integral group ring of a finite Frobenius group G; i.e., that the normalizer of G in the group of units of its integral group ring is $N_{\mathcal{U}}(G) = G \cdot \zeta$, where ζ denotes the centre of the unit group.

MSC: primary 20C05; secondary 16S34, 20C10

1. Introduction

Let $\mathcal{U}(\mathbb{Z}G)$ denote the group of units of the integral group ring $\mathbb{Z}G$ of a finite group G. It is an interesting problem to investigate how G lies in $\mathcal{U}(\mathbb{Z}G)$; i.e., to determine $N_{\mathcal{U}}(G)$, the normalizer of G in $\mathcal{U}(\mathbb{Z}G)$. If ζ denotes the group of central units of $\mathbb{Z}G$, it is clear that $G \cdot \zeta \leqslant N_{\mathcal{U}}(G)$ and it has been conjectured that actually $N_{\mathcal{U}}(G) = G \cdot \zeta$ (this is Problem 43 in S.K. Sehgal's book [10]).

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The first positive result regarding this conjecture is due to D.B. Coleman [2] who proved, in 1964, that the normalizer property holds for modular group algebras of finite p-groups over fields of characteristic p. His methods also apply to prove the conjecture in the case of integral group rings of finite nilpotent groups [10, Theorem 9.1].

More recently, S. Jackowski and Z. Marciniak [5] proved that the property holds for finite groups of odd order as well as for groups which have a normal Sylow 2-subgroup and Y. Li, M.M. Parmenter, and S.K. Sehgal [6] verified that the conjecture is true for the class of finite groups described by N. Blackburn [1]; namely, those groups G such that $R(G)$, the intersection of its non-normal subgroups, is nontrivial.

M. Mazur showed that there is a close relationship between this question and the isomorphism problem, which asks whether finite groups G and H such that $\mathbb{Z}G \cong \mathbb{Z}H$ must be isomorphic (see Mazur [7,8]) and H. Hertweck, trading in this way, proved the existence of groups which would be a negative solution to both the normalizer conjecture and the isomorphism problem (see [3]).

We shall show, in this paper, that the normalizer property holds for the class of Frobenius groups. We recall that a group G is called a *Frobenius group* if it contains a proper nontrivial subgroup H such that $H \cap H^x = 1$ for all $x \in G \setminus H$. Thus, it is clear that Blackburn's groups are not Frobenius and our result enlarges the class of groups for which the normalizer property holds.

2. Preliminary results

We recall some basic facts on Frobenius groups that will be needed in the sequel. For an exhaustive study of the subject, we refer the reader to B. Huppert [4] or D. Passman [9].

Theorem 2.1. *Let G be a Frobenius group and let H be a subgroup such that $H \cap H^g = 1$, for all $g \in G \setminus H$. Write $H^* = H \setminus \{1\}$. Then*:

(a) $K = G \setminus (\bigcup_{x \in G} (H^*)^x)$ *is a characteristic subgroup of* G, $(|H|, |K|) = 1$ *and* $G = K \rtimes H$.
(b) K *is nilpotent.*
(c) *If* $|H|$ *is even, then it contains a unique element* z *of order* 2, *this element is central in* H *and* $z^{-1}kz = k^{-1}$, *for all* $k \in K$. *Furthermore,* K *is abelian.*
(d) *The action of* H^* *on* K *is fixed-point free*; *i.e., if* $h^{-1}kh = k$ *for* $h \in H^*$ *and* $k \in K$, *then* $k = 1$.

It can be shown that the subgroup K in the theorem above is uniquely determined; it is called the *Frobenius kernel* of G. A subgroup H such that $G = K \rtimes H$ is called a *Frobenius complement* of G.

We introduce some notation. For an element $u \in N_{\mathcal{U}}(G)$ we have that $\varphi_u(g) = u^{-1}gu$ defines an automorphism φ_u of G. We shall denote by $\mathrm{Aut}_{\mathcal{U}}(G)$ the group of all these automorphisms. Notice that $\mathrm{Aut}_{\mathcal{U}}(G) = \mathrm{Inn}(G)$ if and only if the normalizer conjecture has a positive answer for G.

Given a Sylow 2-subgroup S of a group G, we denote by I_S the set of all involutions in $\mathrm{Aut}_{\mathcal{U}}(G)$ which keep S pointwise fixed; i.e.,

$$I_S = \left\{\varphi \in \mathrm{Aut}_{\mathcal{U}}(G) \;\middle|\; \varphi^2 = \iota \text{ and } \varphi|_S = \iota\right\}$$

We shall also need the following theorems.

Theorem 2.2 (D.B. Coleman [10, Theorem 9.1]). *Let P be a p-subgroup of a finite group G. For any $\varphi \in \mathrm{Aut}_{\mathcal{U}}(G)$ there exists $\psi \in \mathrm{Inn}(G)$ such that $\psi \circ \varphi(g) = g$ for all $g \in P$.*

Theorem 2.3 (Jackowski and Marciniak [5]). *If $I_S \subseteq \mathrm{Inn}(G)$ for a Sylow 2-subgroup S of G, then $\mathrm{Aut}_{\mathcal{U}}(G) = \mathrm{Inn}(G)$.*

3. The normalizer property for Frobenius groups

We apply the previous results to prove the following theorem.

Theorem 3.1. *Let G be a finite Frobenius group, then the normalizer property holds in the integral group ring $\mathbb{Z}G$.*

Proof. Since the conjecture holds for groups of odd order, we may assume that $|G|$ is even. Write $G = K \rtimes H$ with $(|K|, |H|) = 1$ as in Theorem 2.1. If $2 \mid |K|$, since K is nilpotent (Theorem 2.1(b)), it follows that G has a normal Sylow 2-subgroup and again by [5, Theorem 3.6] the result follows. Therefore, we shall only deal with Frobenius groups G in which the complement H is of even order and hence, by part (c) of Theorem 2.1, the kernel K is abelian. We fix a Sylow 2-subgroup S of H.

Let φ_u be an element of I_S, for some $u \in N_{\mathcal{U}}(G)$. We shall study the action of φ_u on G in several steps.

Claim 1. *For every $g \in G$, we have that $\varphi_u(g)$ and g are conjugate in G.*

In fact, since

$$\varphi_u(g) - g = u^{-1}gu - g = \left[u^{-1}, gu\right] \in [\mathbb{Z}G, \mathbb{Z}G],$$

it follows from [10, Lemma 7.2] that $\varphi_u(g) \sim_G g$.

Claim 2. *Let S_p be a non-identity Sylow p-subgroup of K. Then, either $\varphi_u(k) = k$, for all $k \in S_p$, or $\varphi_u(k) = k^{-1}$, for all $k \in S_p$.*

It follows, from Theorem 2.2, that there exists an element $g_0 \in G$ of the form $g_0 = k_0h_0$ with $k_0 \in K$ and $h_0 \in H$, such that

$$\varphi_u(g)u^{-1}ku = h_0^{-1}k_0^{-1}kk_0h_0 \quad \text{for all } k \in S_p.$$

Since K is abelian, we have that

$$u^{-1}ku = h_0^{-1}kh_0.$$

If $h_0 = 1$, it follows immediately that $\varphi_u(k) = k$, for all $k \in S_p$. Assume $h_0 \neq 1$. As S_p is a characteristic subgroup of K and K is characteristic in G, it follows that the restriction of φ_u to S_p is an automorphism of S_p. Since $\varphi_u \in I_S$, we have that

$$\varphi_u^2(k) = k = h_0^{-2}kh_0^2 \quad \text{for all } k \in S_p.$$

By part (d) of Theorem 2.1 it follows that $h_0^2 = 1$ and, by part (c) of the same theorem, we have that $h_0 = z$, the unique element of H of order 2. In this case $\varphi_u(k) = zkz = k^{-1}$, for all $k \in K$.

Claim 3. *We have that $\varphi_u(k) = k$, for all $k \in K$ or $\varphi_u(k) = k^{-1}$, for all $k \in K$.*

Denote by X the product of all Sylow p-subgroups of K which are pointwise fixed by φ_u and by Y the product of all the others. Then X and Y are characteristic subgroups of G, $K = X \times Y$, $\varphi_u(x) = x$, for all $x \in X$ and $\varphi_u(y) = y^{-1}$, for all $y \in Y$.

For an arbitrary element $k \in K$, we write $k = xy$ with $x \in X$ and $y \in Y$, and we have that

$$u^{-1}xyu = xy^{-1}.$$

Since Claim 1 shows that this implies $xy \sim_G xy^{-1}$, there exists an element $k_0h_0 \in G$ such that

$$xy^{-1} = h_0^{-1}k_0^{-1}xyk_0h_0 = h_0^{-1}xyh_0 = h_0^{-1}xh_0h_0^{-1}yh_0.$$

Thus

$$h_0^{-1}xh_0 = x \quad \text{and} \quad h_0^{-1}yh_0 = y^{-1}.$$

If $h_0 = 1$, then $y^2 = 1$ and, as $2 \nmid |K|$, we have that $y = 1$ for all $y \in Y$. If $h_0 \neq 1$, using again part (d) of Theorem 2.1, it follows that $x = 1$ for all $x \in X$.

We conclude that either $K = X$ or $K = Y$ and the claim follows.

Claim 4. *The complement H is invariant under φ_u; i.e., $\varphi_u(H) \subset H$.*

Since z is a central element of H and φ_u fixes a Sylow 2-subgroup of H, we have that $\varphi_u(z) = u^{-1}zu = z$. Set $h \in H^*$. Once again, by Claim 1, there exists an element $h_0k_0 \in G$ such that

$$u^{-1}hu = k_0^{-1}h_0^{-1}hh_0k_0. \tag{1}$$

Hence

$$zu^{-1}huz = zk_0^{-1}h_0^{-1}hh_0k_0z, \quad \text{so} \quad u^{-1}hu = k_0h_0^{-1}hh_0k_0^{-1}.$$

Thus, comparing with (1), we obtain

$$k_0^{-1}h_0^{-1}hh_0k_0 = k_0h_0^{-1}hh_0k_0^{-1}, \quad \text{so} \quad h_0^{-1}hh_0 = k_0^2h_0^{-1}hh_0k_0^{-2}.$$

Since $h \neq 1$, using again part (d) of Theorem 2.1, we have that $k_0^2 = 1$. As $2 \nmid |K|$, it follows that $k_0 = 1$, and we conclude from (1) that $\varphi_u(h) \in H$, as desired.

Claim 5. *$\varphi_u(h) = h$, for all $h \in H$.*

Let h be an arbitrary element of H. For every element $k \in K$ there exists an element $k_0 \in K$ such that $kh = hk_0$.

Assume first that $\varphi_u(k) = k$ for all $k \in K$. We compute:

$$ku^{-1}huh^{-1}k^{-1} = u^{-1}khuh^{-1}k^{-1} = u^{-1}hk_0uk_0^{-1}h^{-1} = u^{-1}huh^{-1}.$$

As $\varphi_u(H) \subset H$, it follows that $u^{-1}huh^{-1} \in H$ and $ku^{-1}huh^{-1}k^{-1} \in H^k$. Hence

$$u^{-1}huh^{-1} = ku^{-1}huh^{-1}k^{-1} \in H \cap H^k = 1.$$

Consequently, $u^{-1}hu = h$.

Now, assume that $\varphi_u(k) = k^{-1}$, for all $k \in K$. In this case we compute:

$$k^{-1}u^{-1}huh^{-1}k = u^{-1}khuh^{-1}k = u^{-1}hk_0uk_0h^{-1} = u^{-1}huh^{-1}.$$

As before, this implies that $u^{-1}hu = h$.

Finally, we are able to complete the proof. If $u \in N_{\mathcal{U}}(G)$ is such that $\varphi_u \in I_S$, then, for all $h \in H$, we have $\varphi_u(h) = h$ and either $\varphi_u(k) = k$, for all $k \in K$, or $\varphi_u(k) = k^{-1}$, for all $k \in K$. Thus either $\varphi_u(kh) = kh$, for all kh in G, or $\varphi_u(kh) = k^{-1}h = zkhz$, for all kh in G. In both cases $\varphi_u \in \mathrm{Inn}(G)$ and it follows from Theorem 2.3 that $\mathrm{Aut}_{\mathcal{U}}(G) = \mathrm{Inn}(G)$ or, equivalently, that the normalizer property holds in G. $\square$

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